More on Definite Integrals Class 37

In this worksheet, we aim to evaluate $\int_a^b \frac{1}{x}\,\mathrm{d}x$ for any a, b possible.

1. This problem studies definite integrals of the form

$$g_b(t) = \int_t^{bt} \frac{1}{x}\,\mathrm{d}x,$$

for $b > 1$, $t > 0$. Throughout this problem, you should use central sums to estimate definite integrals. You need not use a high n value; $n = 50$ is plenty, and $n = 5$ gives pretty good accuracy.

   (a) For $b = 4$, the above definite integral can be easily estimated on the spreadsheet by entering `1/x` for f(x), a value of t (maybe start with $t = 1$) for xmin, and `=4*C2` for xmax. The number 4 in the formula for xmax is b, and you can try changing it later.

   By increasing n, get an estimate for $g_4(1) = \int_1^4 \frac{1}{x}\,\mathrm{d}x$.

   (b) Try varying $t > 0$, but keep $b = 4$. Obtain estimates for $g_4(t)$.

   (c) Now set $t = 1$, and vary $b > 1$. Obtain estimates for $g_b(1)$.

   (d) Can you find b, t, s such that $g_b(t) \neq g_b(s)$?

   (e) Find b such that $g_b(t) = 1$ for all t.

   (f) Find $g_{e^c}(t)$ for a few values of $c > 0$, $t > 0$ and describe the rule.

   (g) Hence give a formula for $\int_t^{bt} \frac{1}{x}\,\mathrm{d}x$.

2. In the last problem, we discovered $\int_1^b \frac{1}{x}\,\mathrm{d}x = \ln(b)$. Assume $b > a > 0$ in this problem.

   (a) Use the definite integral theorems to explain why $\int_a^b \frac{1}{x}\,\mathrm{d}x = \ln(b) - \ln(a)$.

   (b) Use the Fundamental Theorem of Calculus to explain why $\int_a^b \frac{1}{x}\,\mathrm{d}x = \ln(b) - \ln(a)$.

   (c) Use a sketch and the result of part (b) to explain why $\int_{-a}^{-b} \frac{1}{x}\,\mathrm{d}x = \ln(b) - \ln(a)$.

So far, we have learned $\int_a^b \frac{1}{x}\,\mathrm{d}x = \ln|b| - \ln|a|$ provided either $a < b < 0$ or $0 < a < b$. In the final problem, we will study the other case: $a < 0 < b$.

3. (a) Sketch $y = \dfrac{1}{x}$ and use your sketch to estimate $\displaystyle\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$.

(b) Enter $\frac{1}{x}$ into the spreadsheet as `IF(x=0,0,1/x)`. Use left, right and central sums to estimate $\displaystyle\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$. Does this confirm your estimate from (a)?

Note: the `IF` statement is there to fill in a value for `1/x` at `x` $= 0$. Without it, the spreadsheet would be confused.

On page 285 of the textbook, *More General Riemann Sums*, it is noted that we need not use just the left or right Riemann sums to estimate an integral. Consider the following construction of a Riemann sum with $3n$ terms used to estimate $\displaystyle\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$

- The first n terms correspond to left rectangles of equal width, which together estimate the signed area under $\frac{1}{x}$ for x between -1 and 0.
- The remaining $2n$ terms correspond to right rectangles of equal width (but different width to the left rectangles), which together estimate the signed area under $\frac{1}{x}$ for x between 0 and 1.

This Riemann sum should have the same limit (as $n \to \infty$) as the Riemann sums evaluated in part (b).

(c) Make a new sketch of $y = \dfrac{1}{x}$. Draw rectangles on your sketch whose total area represents the Riemann sum described above, with $n = 3$ (so there should be $3n = 9$ rectangles in total).

(d) Using Σ-notation, give a formula for the Riemann sum described above. Simplify your formula to show that the Riemann sum is $\displaystyle\sum_{j=1}^{n} \frac{1}{2j(2j-1)}$.

(e) Explain why the Riemann sum is positive and increasing as n increases.

Note: In fact, $\lim_{n\to\infty}$(Riemann sum described above with $3n$ terms) $= \ln 2$, but you do not need to use this to answer the question.

(f) What can we conclude about $\displaystyle\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$?